Module 6
Columns and Struts

Columns and Struts

- Any member subjected to axial compressive load is called a column or Strut.
- A vertical member subjected to axial compressive load – *COLUMN* (Eg: Pillars of a building)
- An inclined member subjected to axial compressive load - *STRUT*
- A strut may also be a horizontal member
- Load carrying capacity of a compression member depends not only on its cross sectional area, but also on its length and the manner in which the ends of a column are held.

- Equilibrium of a column – Stable, Unstable, Neutral.
- Critical or Crippling or Buckling load – Load at which buckling starts
- Column is said to have developed an elastic instability.

Classification of Columns

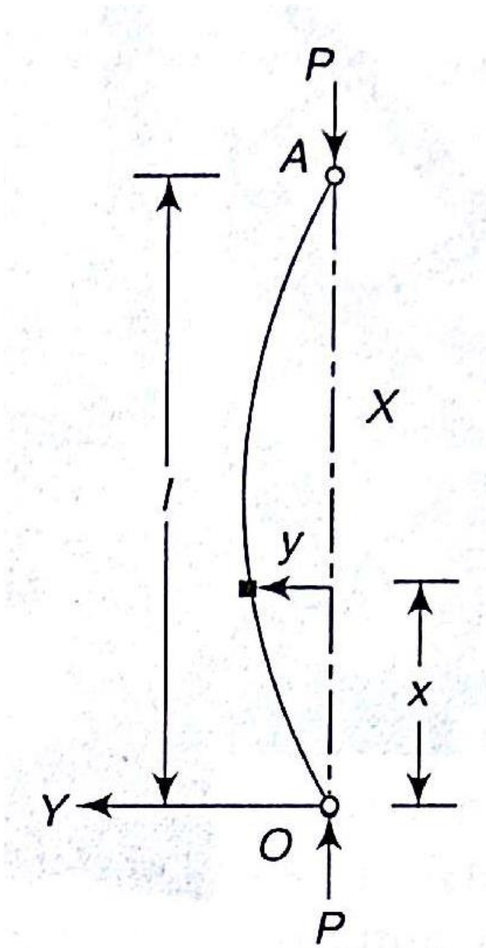
- According to nature of failure – short, medium and long columns
- 1. Short column – whose length is so related to its c/s area that *failure occurs mainly due to direct compressive stress* only and the role of bending stress is negligible
- 2. Medium Column - whose length is so related to its c/s area that *failure occurs by a combination of direct compressive stress and bending stress*
- 3. Long Column - whose length is so related to its c/s area that *failure occurs mainly due to bending stress* and the role of direct compressive stress is negligible

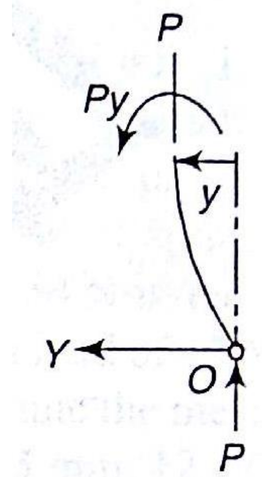
Euler's Theory

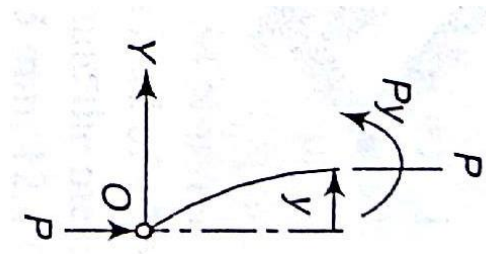
- Columns and struts which fail by buckling may be analyzed by Euler's theory
- <u>Assumptions made</u>

- the column is initially straight
- the cross-section is uniform throughout
- the line of thrust coincides exactly with the axis of the column
- the material is homogeneous and isotropic
- the shortening of column due to axial compression is negligible.

Case (i) Both Ends Hinged







$$EI\,\frac{d^2 y}{dx^2} = M = -Py$$

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The equation can be written as $\frac{d^2y}{dx^2} + \alpha^2 y = 0$ where $\alpha^2 = \frac{P}{EI}$

The solution is $y = A\sin\alpha x + B\cos\alpha x$

At $x = 0, y = 0, \therefore B = 0$

at $x = l, y = 0$ and thus $A\sin\alpha l = 0$

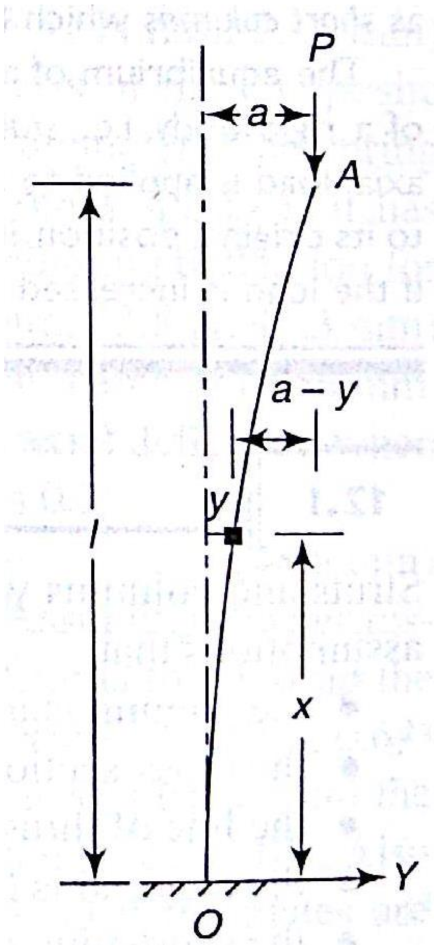
If $A = 0$, y is zero for all values of load and there is no bending.

$\therefore \quad \sin\alpha l = 0 \quad$ or $\quad \alpha l = \pi \quad$ (considering the least value)

or $\quad \alpha = \pi / l$

$\therefore \quad$ Euler crippling load, $P_e = \alpha^2 EI = \frac{\pi^2 EI}{l^2}$

Case (ii) One end fixed other free



$$EI\frac{d^2y}{dx^2} = M = P(a - y) = Pa - Py$$

$$EI\frac{d^2y}{dx^2} = M = P(a - y) = Pa - Py$$

$$\frac{d^2y}{dx^2} + \alpha^2 y = \frac{P \cdot a}{EI} \quad \text{where} \quad \alpha^2 = \frac{P}{EI}$$

The solution is $y = A\sin\alpha x + B\cos\alpha x + \dfrac{P.a}{EI\alpha^2}$

$$= A\sin\alpha x + B\cos\alpha x + a$$

$$x = 0, y = 0, \ \therefore B = -a;$$

$$x = 0, \ \frac{dy}{dx} = 0$$

or $\quad A\alpha\cos\alpha x - B\alpha\sin\alpha x = 0 \quad$ or $\quad A = 0$

$$y = -a\cos\alpha x + a = a(1 - \cos\alpha x)$$

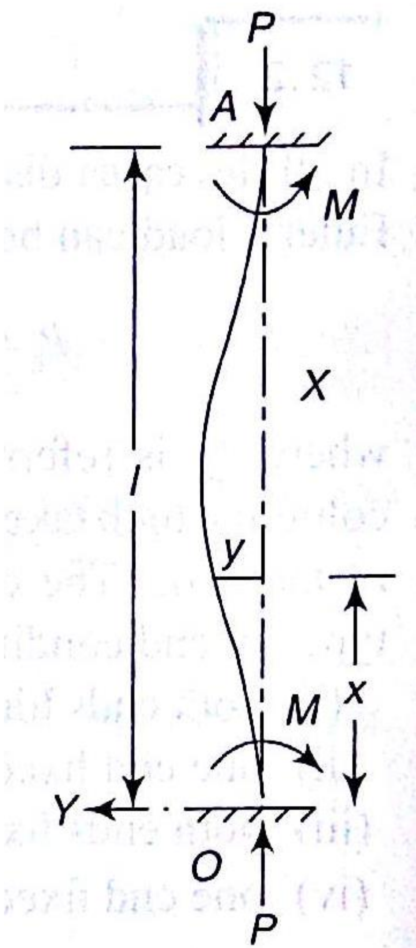
At $x = l, y = a, \therefore a = a(1 - \cos \alpha l)$

or $\cos \alpha l = 0$ or $\alpha l = \frac{\pi}{2}$ (considering the least value)

$$\alpha = \pi / 2l$$

$\therefore$ Euler crippling load, $P_e = \alpha^2 EI = \frac{\pi^2 EI}{4l^2}$

Case (iii) Fixed at both ends



$$EI\frac{d^2y}{dx^2} = -Py + M$$

$$EI\frac{d^2y}{dx^2} = -Py + M$$

$$\frac{d^2y}{dx^2} + \alpha^2 y = \frac{M}{EI} \quad \text{where} \quad \alpha^2 = \frac{P}{EI}$$

The solution is $y = A\sin\alpha x + B\cos\alpha x + \dfrac{M}{EI\alpha^2} = A\sin\alpha x + B\cos\alpha x + \dfrac{M}{P}$

$$x = 0, y = 0, \therefore\ B = -\frac{M}{P};$$

$$x = 0, \frac{dy}{dx} = 0$$

or $\quad A\alpha\cos\alpha x - B\alpha\sin\alpha x = 0 \quad$ or $\quad A = 0$

$$\therefore\ y = -\frac{M}{P}\cos\alpha x + \frac{M}{P} = \frac{M}{P}(1 - \cos\alpha x)$$

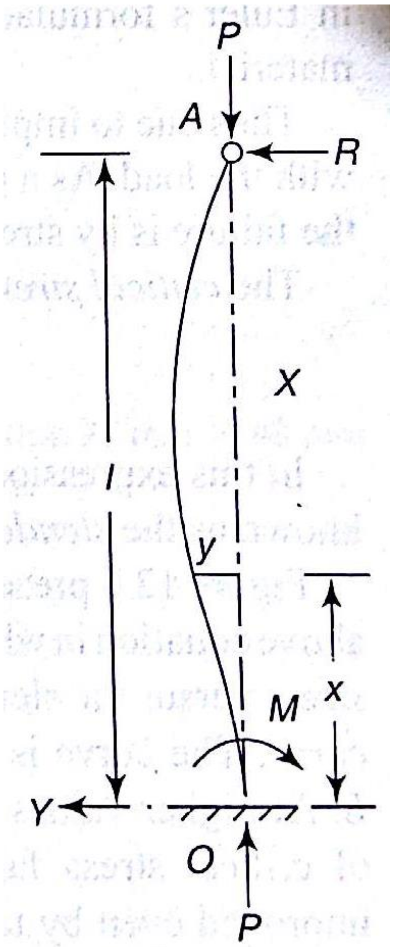
At $x = l, y = 0, \therefore\ 0 = \frac{M}{P}(1 - \cos \alpha l)$ or $\cos \alpha l = 1$

or $\alpha l = 2\pi$ (considering the least value)

or $\alpha = 2\pi / l$

$$\therefore \text{ Euler crippling load, } P_e = \alpha^2 EI = \frac{4\pi^2 EI}{l^2}$$

Case (iv) One end fixed, other hinged



$$EI\frac{d^2y}{dx^2} = -Py + R(l - x)$$

$$EI\frac{d^2y}{dx^2} = -Py + R(l-x)$$

$$\frac{d^2y}{dx^2} + \alpha^2 y = \frac{R(l-x)}{EI} \quad \text{where} \quad \alpha^2 = \frac{P}{EI}$$

The solution is $y = A\sin\alpha x + B\cos\alpha x + \dfrac{R(l-x)}{EI\alpha^2}$

$$= A\sin\alpha x + B\cos\alpha x + \frac{R}{P}(l-x)$$

At $x = 0, y = 0, \therefore\ B = -\dfrac{Rl}{P}$;

At $x = 0,\ \dfrac{dy}{dx} = 0$

or $\quad A\alpha\cos\alpha x - B\alpha\sin\alpha x - \dfrac{R}{P} = 0$

or $\quad A = \dfrac{R}{P\alpha}$

$$\therefore \quad y = \frac{R}{P\alpha}\sin \alpha x - \frac{Rl}{P}\cos \alpha x + \frac{R}{P}(l - x)$$

At $x = l, y = 0, \therefore \; 0 = \frac{R}{P\alpha}\sin \alpha l - \frac{Rl}{P}\cos \alpha l$

or $\quad \tan \alpha l = \alpha l$

$\alpha l = 4.49$ rad (considering the least value)

$\alpha = 4.49 / l$

$$\therefore \text{ Euler crippling load, } P_e = \alpha^2 EI = \frac{4.49^2 EI}{l^2} = \frac{20.2 EI}{l^2} \approx \frac{2\pi^2 EI}{l^2}$$

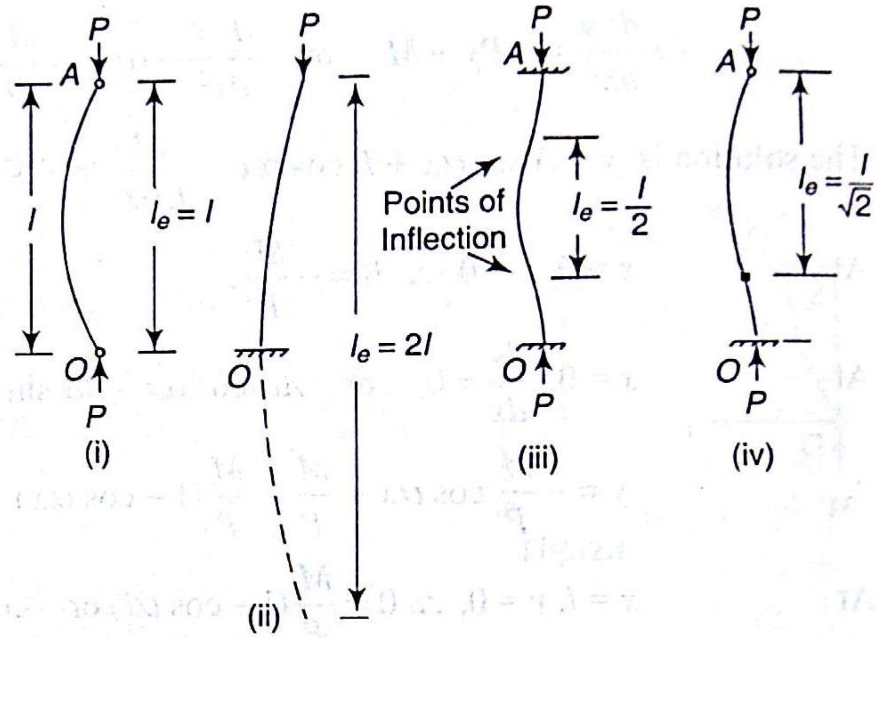
Equivalent Length (l_e)

Euler's load can be expressed as $P_e = \frac{\pi^2 EI}{l_e^2}$

where l_e^2 is referred as *equivalent length* of the column which takes into account the type of fixing of the ends.

The equivalent lengths for different types of end conditions are

(i) both ends hinged, $l_e = l$
(ii) one end fixed and the other free, $l_e = 2l$
(iii) both ends fixed, $l_e = l/2$
(iv) one end fixed, other hinged, $l_e = l/\sqrt{2}$



Limitations of Euler’s Formula

- Assumption – Struts are initially perfectly straight and the load is exactly axial.
- There is always some eccentricity and initial curvature present.
- In practice a strut suffers a deflection before the Crippling load.

- Critical stress (σ_c) – average stress over the cross section

$$\sigma_c = \frac{P_e}{A} = \frac{\pi^2 EI}{Al_e^2}$$

$$= \frac{\pi^2 EAk^2}{Al_e^2}$$

$$\sigma_c = \frac{\pi^2 E}{(l_e/k)^2}$$

- l/k is known as **Slenderness Ratio**

Slenderness Ratio

- **Slenderness ratio** is the ratio of the length of a column and the radius of gyration of its cross section
- Slenderness Ratio = l/k

The Radius of Gyration $\mathbf{k_x}$ of an Area (A) about an axis (x) is defined as:

$$I_x = k_x^{\,2} A$$

$$\mathbf{k_x} = \sqrt{\frac{I_x}{A}}$$

Rankine's Formula
OR
Rankine-Gorden Formula

- Euler's formula is applicable to long columns only for which l/k ratio is larger than a particular value.
- Also doesn't take in to account the direct compressive stress.
- Thus for columns of medium length it doesn't provide accurate results.
- Rankine forwarded an empirical relation

$$\frac{1}{P} = \frac{1}{P_c} + \frac{1}{P_e}$$

where P = Rankine's crippling load

P_c = ultimate load for a strut = $\sigma_u \cdot A$, constant for a material

P_e = Eulerial load for a strut = $\pi^2 EI/l^2$

- For short columns, P_e is very large and therefore $1/P_e$ is small in comparison to $1/P_c$. Thus the crippling load P is practically equal to P_c
- For long columns, P_e is very small and therefore $1/P_e$ is quite large in comparison to $1/P_c$. Thus the crippling load P is practically equal to P_e

$$\frac{1}{P} = \frac{1}{P_c} + \frac{1}{P_e}$$

$$\frac{1}{P} = \frac{P_e + P_c}{P_c P_e}$$

$$P = \frac{P_c P_e}{P_e + P_c} = \frac{P_c}{1 + P_c / P_e} = \frac{\sigma_c \cdot A}{1 + \frac{\sigma_c A \cdot l^2}{\pi^2 EI}}$$

$$= \frac{\sigma_c \cdot A}{1 + \frac{\sigma_c \cdot A \cdot l^2}{\pi^2 EAk^2}}$$

$$P = \frac{\sigma_c \cdot A}{1 + a\left(\frac{l}{k}\right)^2}$$

where σ_c is the crushing stress

a is the Rankine's constant $(\sigma_c/\pi^2 E)$

$$\frac{1}{P} = \frac{1}{P_c} + \frac{1}{P_e}$$

$$\frac{1}{P} = \frac{P_e + P_c}{P_c P_e}$$

$$P = \frac{P_c P_e}{P_e + P_c} = \frac{P_c}{1 + P_c / P_e} = \frac{\sigma_c \cdot A}{1 + \frac{\sigma_c A \cdot l^2}{\pi^2 EI}}$$

$$= \frac{\sigma_c \cdot A}{1 + \frac{\sigma_c \cdot A \cdot l^2}{\pi^2 EAk^2}}$$

$$P = \frac{\sigma_c \cdot A}{1 + a\left(\frac{l}{k}\right)^2}$$

where σ_c is the crushing stress

a is the Rankine's constant $(\sigma_c/\pi^2 E)$

- A Factor of Safety may be considered for the value of σ_c in the above formula

- Rankine's formula for columns with other end conditions

$$P = \frac{\sigma_c \cdot A}{1 + a\left(\frac{l_e}{k}\right)^2}$$